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Three dimensional $N = 2$ supersymmetry on the lattice

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Abstract: We show how 3-dimensional, $N = 2$ supersymmetric theories, including super QCD with matter fields, can be put on the lattice with existing techniques, in a way which will recover supersymmetry in the small lattice spacing limit. Residual supersymmetry breaking effects are suppressed in the small lattice spacing limit by at least one power of the lattice spacing a.

Keywords: Lattice Quantum Field Theory, Nonperturbative Effects, Field Theories in Lower Dimensions, Supersymmetric gauge theory.

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1. Introduction

Supersymmetry is a beautiful and phenomenologically interesting candidate for phys-ics beyond the standard model. Our understanding of quantum field theory has been improved by our ability to make exact statements within supersymmetric theories, such as non-renormalization theorems and dualities.

There is a large class of nonperturbative results or conjectured results for supersymmetric theories, particularly supersymmetric Yang-Mills theories, both in four dimensions [1 – 3] and in three dimensions [4 – 7]. For instance, in 4 dimensional $N = 2$ super Yang-Mills (SYM) theory, Seiberg and Witten's work [2] essentially solves the theory. In three dimensions, a number of exquisite results have been obtained involving mirror symmetry in $N = 4$ SYM theory [4] and also in $N = 2$ theories [5, 6]. In [7] these symmetries were motivated in the context of string theory. There is also a beautiful conjecture by Witten [8] which makes $N = 1$ SYM theory in 3 dimensions an excellent testbed for spontaneous (dynamical) supersymmetry breaking.

It would be very helpful if we were able to test some of the techniques for studying supersymmetric theories, for instance by "solving" the theories involved in a non-perturbative way. The lattice is the best candidate method for solving such theories. Unfortunately, the lattice regulator almost inevitably breaks the supersymmetry. The lattice is, after all, a regularization scheme designed to preserve exact gauge symmetry at the expense of manifest Poincaré invariance. Since supersymmetry is a space-time symmetry, i.e. an extension of the Poincaré group, it is of no surprise that it is broken on the lattice.

It is certainly possible to use the lattice to study a theory which possesses a symmetry broken by the lattice action. After all, the main application of the lattice technique is the study of QCD, which has O(4) euclidean invariance, broken to (hyper)cubic symmetry on the lattice. In this case, the symmetry is recovered automatically, up to corrections suppressed by the lattice spacing, because the low-energy phenomena are described by an effective theory which possesses full $O(4)$ symmetry accidentally. That is, the low dimension operators which respect (hyper)cubic symmetry also respect $O(4)$ symmetry, so that symmetry is recovered in the infrared. Unfortunately, for supersymmetric theories containing scalars, the scalar mass term is low dimensional and breaks the supersymmetry. Therefore very fine tuning of the lattice action is generally required to recover supersymmetry in the infrared.

A great deal of work has gone into looking for a way to preserve supersymmetry, at least in the small lattice spacing limit [9 − 13]. In 4 dimensions, it appears that one should be able to implement $N = 1$ SYM theory without matter [14 − 16], that is, a theory containing only gauge fields and an adjoint representation, Weyl fermion (the gluino). This is because this case does not involve a scalar field, so supersymmetry is an accidental symmetry once enough chiral symmetry is present to prevent a mass for the gluino. This is now possible with recent advances in 4D chiral fermions [17 − 20]. For cases involving scalar fields, most attempts involve building in enough supersymmetry into the lattice construction to ensure that scalar mass renormalization does not occur. This is true, for instance, of proposed implementations of the Wess-Zumino model in 2 dimensions [9, 10, 21]. Various 2 and 3 dimensional, highly supersymmetric pure Yang-Mills theories can be implemented by a technique developed by Kaplan, Katz, and Unsal [12, 22] (see however [23]). A technique for putting $N = 2$ SYM theory on the lattice using "twisted" supersymmetries with Kähler-Dirac fermions has been developed by Catterall [24] and recently extended to $N = 4$ supersymmetry in 4 dimensions [25]. Also D'Adda and collaborators [26] have recently completed the construction of a 2-d lattice which retains *all* the supersymmetries of the target SYM theory exactly on the lattice using twisted supersymmetry and "fermionic" links where supercharges live in direct analogy to gauge fields living on bosonic links. This would seem to be well motivated by analogy to the continuous superspace formulation of supersymmetry in which the generators of the supersymmetry are regarded as translation operators along new Grassman valued coordinates.

These techniques often come with certain limitations such as confinement to even dimensions, to very highly supersymmetric theories or to specific numbers of fermionic fields. They are frequently made difficult by highly non-trivial lattice actions that can contain a wide range of complications such as non-commutativity, the failure of reflection positivity, unphysical moduli, or sign problems with complex fermion determinants. Most of these issues can or have been overcome to varying extents, yet they still remain to complicate the treatment.

We argue here that, for all the 3-dimensional theories of interest with $N = 2$ (4 real supercharges) or more, one can proceed by actually doing the fine tuning on the most conventional lattice action available, the Wilson action with Wilson fermions. The reason this approach is available is because these theories are super-renormalizable. That means that loop corrections involving the ultraviolet converge in powers of the lattice spacing, so the tuning only requires a lattice perturbation theory calculation to a finite loop order, which turns out to be 2 loops. Further, we explicitly do this calculation for the case of

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SYM theory plus fundamental matter. This holds out the possibility of testing some very interesting claims [5, 6] of exact results in $N = 2$ SYM theory with matter. Some similarly spirited work in the context of 2-D $N = 2$ pure SYM has recently appeared in [27].

In section 2, we will explain the general idea which makes it possible to preserve supersymmetry, in the small lattice spacing limit, in 3-dimensional, $N = 2$ supersymmetric theories. In section 3 we present in detail the implementation of the Wess-Zumino model, that is, a theory of fermions and scalars. In section 4 we present the implementation of gauge theories. Finally, some concluding remarks appear in section 5.

2. Super-renormalizability and SUSY breaking

No one is interested in the results of a lattice calculation *per se*. After all, a lattice "field theory" is actually just a statistical model, not a true field theory. The reason that the lattice technique can teach us something about field theory, is that in the *infrared*, the correct effective description of a lattice theory is as a quantum field theory. What one must do is to ensure that the infrared behavior of the lattice theory coincides with the continuum quantum field theory of interest, or at least that it does so in the small lattice spacing limit, so that the behavior of the field theory can be probed by making a zero lattice spacing extrapolation.

It is easy to write down a lattice gauge theory which, at tree level, will look in the infrared like the theory of interest.[1] The problem is that, in the UV (at the lattice spacing scale), the lattice theory typically does not have the full symmetries of the theory we are interested in. Generally it is possible to formulate lattice theories so that they have exact gauge and (hyper)cubic symmetries. However, under supersymmetry, the variation of a fermionic field can involve the derivative of a bosonic field; and since derivatives become finite differences on the lattice, supersymmetry will generically be badly broken at the lattice spacing scale. Furthermore, even if we construct the lattice theory to satisfy supersymmetric relations in the infrared, radiative effects involving UV (SUSY breaking) modes will typically communicate those effects to the infrared modes of interest.

The IR effective theory is not the tree level theory. Rather, it is the theory one obtains, by writing down the most general continuum quantum field theory consistent with the field content and symmetries of the lattice, and performing a matching calculation between the lattice theory and that continuum effective theory, to determine what the actual parameters of the IR effective theory are. For instance, if we made a tree level lattice implementation of the Wess-Zumino model (which in 3 dimensions exhibits $N = 2$ supersymmetry, that is,

[1] In four dimensions this statement is true of vectorlike theories, but serious complications arise if one wants a chiral theory, that is, a theory with two component spinors in a representation which is not real or pseudoreal and which are not balanced by an equal number of spinors of opposite handedness in the same representation. To date, it is understood how, in principle, to put certain abelian chiral theories on the lattice; it is not clear, even in principle, how to put more general theories on the lattice while still retaining manifest gauge invariance [28]. Fortunately, for the three dimensional theories of interest to us, this is not relevant, because chiral fermions do not exist in odd dimensions.

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it has 4 real supersymmetry generators),

$$\mathcal{L}_{\rm bare} = \partial_\mu \Phi^* \partial^\mu \Phi + \psi^\dagger \not{\partial} \psi + \Big(\lambda \Phi \psi^\top e \psi + \text{h.c.} \Big) + \lambda^2 \Big(\Phi^\dagger \Phi \Big)^2 , \tag{2.1}$$

with Φ a complex scalar and ψ a two component spinor, then we would generically recover an infrared theory where all terms permissible with this field content were present;

$$\begin{aligned} \mathcal{L}_{\rm IR} = & \; Z_\phi \partial_\mu \Phi^* \partial^\mu \Phi + Z_\psi \psi^\dagger \not{\partial} \psi + m_\phi^2 \Phi^* \Phi + m_\psi \psi^\dagger \psi + \\ & + \Big(\lambda_y \Phi \psi^\top e \psi + \text{h.c.} \Big) + \lambda_s^2 \Big(\Phi^* \Phi \Big)^2 + \text{ (High Dim.)} . \end{aligned} \tag{2.2}$$

Here e is the antisymmetric 2×2 matrix, $e = i\sigma^2$. Z_ϕ and Z_ψ represent the difference in field normalization between the lattice and continuum fields; they can be removed by a field rescaling, but we must keep them in mind when we compare lattice correlation functions with their continuum counterparts. The point is that the IR behavior typically involves radiatively generated terms which do not respect the intended supersymmetry. In particular one does not expect $m_\psi^2 = m_\phi^2$.

In 4 dimensions this problem is severe. The SUSY violating, radiatively induced terms appear at all orders in perturbation theory, with coefficients, at high order, which are only suppressed with respect to the lower order coefficients by powers of a dimensionless coupling. Further, additive scalar mass renormalizations are divergently large at *every loop order*. That is, in 4 dimensions, the contributions to the mass squared parameter are of order

$$\delta m^2 \text{ at} \qquad 1 \text{ loop: } \frac{\lambda^2}{a^2}\,; \quad 2 \text{ loops: } \frac{\lambda^4}{a^2}\,; \quad 3 \text{ loops: } \frac{\lambda^6}{a^2}\,; \; \ldots \,, \tag{2.3}$$

where a is the lattice spacing. Every such coefficient is problematic; a severe nonperturbative tuning is needed to remove them. It is not at all clear how to perform such a tuning; generally we can only perform nonperturbative tunings in lattice gauge theories if we have one exact conservation law or Ward identity per tuning required.

The beauty of 3D is that the desired theory is generally super-renormalizable. Consequently, the UV is very weakly coupled; specifically, as the lattice spacing is taken to zero, the coupling at the scale of the lattice spacing falls linearly with lattice spacing a. This means that, while the SUSY breaking nature of the UV regulator radiatively induces SUSY breaking effects in the IR, the matching calculation which determines them converges very quickly. At each loop order, we determine the matching of parameters to one more power of the lattice spacing a. For instance, in the above model, if we compute the mass squared for the scalar field, generated by UV physics, the contributions at different orders in the loopwise expansion are again of order λ^2, λ^4, λ^6, ... But λ^2 has mass dimension 1. Since the matching calculation involves only UV physics, the only scale which can balance the explicit powers of mass is the lattice spacing scale. Therefore, the terms in the loopwise expansion are of order

$$\delta m^2 \text{ at} \qquad 1 \text{ loop: } \frac{\lambda^2}{a}\,; \quad 2 \text{ loops: } \lambda^4\,; \quad 3 \text{ loops: } a\lambda^6\,; \; \ldots \,. \tag{2.4}$$

The one and two loop contributions are significant and must be removed by an appropriate

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counterterm. However, three and higher loop effects vanish in the $a \to 0$ limit, and so can be neglected. For the scalar self-coupling λ_s^2, the one loop correction is already $\mathrm{O}(a\lambda^4)$, and so a tree level treatment is already sufficient.

We should stress here that the infrared cancellation which ensures that only the scale a can appear to balance the explicit mass dimensions in the coupling, is a generic property of matching calculations in effective field theory and contains no statement about the IR physics of supersymmetry. Infrared divergences arise from large length scale (or low loop momentum) behavior. By construction the two theories for which any matching in an effective field theory formulation is being performed have the same IR behavior, and so any IR divergence will cancel in the difference. The statement is then simply that the IR extension of the lattice theory does describe the SUSY theory of interest provided that it contains the same degrees of freedom and that the coefficients of the terms in the lagrangian can be matched with those of the theory of interest. In the 4-d case this matching does not seem possible within a perturbative framework; however, in a super-renormalizable theory it certainly is.

We see that only a finite loop order is needed before all remaining corrections are suppressed by powers of a. It is therefore feasible to perform the matching calculation to the requisite order analytically, and to tune the lattice theory based on the *purely analytic* result of this perturbative matching calculation, to ensure that the IR effective theory satisfies all relations implied by SUSY up to a suppressed corrections.

How hard is this tuning? In the present paper we will be satisfied with removing SUSY violating effects which do not vanish with the lattice spacing; that is, we will leave $\mathrm{O}(a)$ SUSY violating effects, but prevent $\mathrm{O}(a^0)$ SUSY violation. In this case it is straightforward to show that the scalar masses must be determined at two loops, and the fermionic masses must be determined at one loop. All other lagrangian parameters may be taken at tree level. This same technique has been applied to purely bosonic 3-dimensional Yang-Mills Higgs theory, in the context of understanding the electroweak phase transition, in [29 – 33]. Note that we will not attempt to provide renormalized operators, and in particular we will not compute the SUSY violating renormalization of the vacuum energy, which would require a 4 loop effort.

To conduct the matching calculation, we need to compute some set of correlation functions in both the lattice theory and the continuum, supersymmetric theory, and equate the answers. At the level of interest, we need only do so for two correlation functions, which must be sensitive to the scalar and fermionic masses. The obvious candidates are the respective two-point functions (self-energies). Further, there is no need to compute the supersymmetric values, since we already know that they vanish identically. Therefore, all that is required is to compute the fermionic self-energy at zero momentum, at one loop, and the scalar self-energy at zero momentum, at two loops, and to assign counterterms to cancel these contributions. Provided that we take suitable combinations of loop contributions (fermionic and bosonic), we are guaranteed that all loop integrations will be IR finite-precisely because supersymmetry is satisfied in the infrared. Therefore, the matching calculation will consist of computing a handful of IR finite linear combinations of Feynman graphs involving scalars, fermions, and gauge fields.

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In our calculations here, we will eliminate all corrections which are unsuppressed by powers of the lattice spacing a; but corrections suppressed by a single power of a will still exist. If we were very strong, it would be possible to eliminate $\mathrm{O}(a)$ errors as well, by conducting a 1 loop determination of couplings, a 2 loop determination of fermionic masses, and a 3 loop determination of scalar masses. (Tree level improvement of the fermionic action would also be required.) Such a calculation has been carried out for three dimensional, $\mathrm{O}(N)$ symmetric scalar field theories [34]; indeed, in this case many of the $\mathrm{O}(a^2)$ corrections have also been eliminated [35]. In principle it is possible to eliminate errors to any fixed order in a, by using a suitably tree-level improved action and evaluating a number of low dimension operators to suitable loop levels; as the number of powers of a accuracy desired increases, more and more operators must be improved. We will not attempt such a program here.

3. Detailed treatment of the Wess-Zumino model

Consider a theory of scalars and fermions with $N = 2$ supersymmetry in 3 dimensions. The matter content is the same as for a 4 dimensional $N = 1$ supersymmetric theory. To make the extension to gauge theories simpler, we consider a theory with a *global* $\mathrm{SU}(N_{\mathrm{c}})$ symmetry, and three sets of fields; those in the fundamental representation, Φ_f, ψ_f; those in the antifundamental representation, Φ_a, ψ_a; and those in the singlet representation, Φ_s, ψ_s.

The most general superpotential, excluding mass terms, is

$$W = \lambda_{ijk}\Phi_{f,i}\Phi_{a,j}\Phi_{s,k} + \frac{\xi_{ijk}}{6}\Phi_{s,i}\Phi_{s,j}\Phi_{s,k}\,, \tag{3.1}$$

with ξ_{ijk} totally symmetric in its indices. We have left out supersymmetric mass terms simply because including them will not lead to any change in the mass counterterms we will need; any loopwise correction involving a mass term will be $\mathrm{O}(m^2)$; at worst this leads to $\mathrm{O}(m^2\lambda^2 a)$ effects, which are beyond the order under consideration.

The lagrangian which follows from this superpotential is

$$\begin{aligned}
\mathcal{L} = {} & \Big[(\partial_\mu\Phi_{f,i})^*(\partial^\mu\Phi_{f,i}) + \psi^\dagger_{f,i}\,\partial\!\!\!/\,\psi_{f,i} + (f\to a,s)\Big] + \lambda^*_{ijk}\lambda_{ilm}\Phi^*_{a,j}\cdot\Phi_{a,l}\Phi^*_{s,k}\Phi_{s,m} + \\
& + \lambda^*_{jik}\lambda_{lim}\Phi^*_{f,j}\cdot\Phi_{f,l}\Phi^*_{s,k}\Phi_{s,m} + \lambda^*_{jki}\lambda_{lmi}\Phi^*_{f,j}\cdot\Phi^*_{a,k}\Phi_{f,l}\cdot\Phi_{a,m} + \\
& + \left(\frac{\lambda_{jki}\xi^*_{lmi}}{2}\Phi_{f,j}\cdot\Phi_{a,k}\Phi^*_{s,l}\Phi^*_{s,m} \;+\mathrm{h.c.}\right) + \frac{\xi_{ijk}\xi^*_{ilm}}{4}\Phi_{s,j}\Phi_{s,k}\Phi^*_{s,l}\Phi^*_{s,m} + \\
& + \bigg(\lambda_{ijk}\Big[\Phi_{s,k}\psi^\top_{f,i}\,e\,\psi_{a,j} + \Phi_{f,i}\psi^\top_{s,k}\,e\,\psi_{a,j} + \Phi_{a,j}\psi^\top_{s,k}\,e\,\psi_{f,i}\Big] + \\
& \qquad + \lambda^*_{ijk}\Big[\Phi^*_{s,k}\psi^\dagger_{f,i}\,e\,\psi^*_{a,j} + \Phi^*_{f,i}\psi^\dagger_{s,k}\,e\,\psi^*_{a,j} + \Phi^*_{a,j}\psi^\dagger_{s,k}\,e\,\psi^*_{f,i}\Big]\bigg) + \\
& + \left(\frac{\xi_{ijk}}{2}\Phi_{s,i}\psi^\top_{s,j}\,e\,\psi_{s,k} + \frac{\xi^*_{ijk}}{2}\Phi^*_{s,i}\psi^\dagger_{s,j}\,e\,\psi^*_{s,k}\right),
\end{aligned} \tag{3.2}$$

where the dot product $\cdot$ refers to $\mathrm{SU}(N_{\mathrm{c}})$ indices. Note that the 3-dimensional gamma matrices are 2×2 matrices, and are in fact nothing but the Pauli matrices, σ^μ.

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The lattice implementation of this theory is obtained by discretizing space onto a cubic lattice of spacing a, replacing $\int d^3x\mathcal{L}$ with $a^3\sum_x \mathcal{L}$, and writing the gradient terms as follows:

$$\int d^3x\, (\partial_\mu\Phi)^*(\partial^\mu\Phi) \to a^3\sum_{x,\mu}\left(\frac{\Phi^*(x+a\hat{\mu})-\Phi^*(x)}{a}\;\frac{\Phi(x+a\hat{\mu})-\Phi(x)}{a}\right), \tag{3.3}$$

$$\begin{aligned}\int d^3x\, \psi^\dagger \partial\!\!\!/\psi \to a^3\sum_{x,\mu}\psi^\dagger\Bigg[\sigma^\mu\frac{\psi(x+a\hat{\mu})-\psi(x-a\hat{\mu})}{2a}+\\ +\frac{ar}{2}\,\frac{-\psi(x+a\hat{\mu})+2\psi(x)-\psi(x-a\hat{\mu})}{a^2}\Bigg].\end{aligned} \tag{3.4}$$

The "extra" term in the fermion gradient term, which for slowly varying fields looks like $-(ra/2)\psi^\dagger\partial^2\psi$, is called the Wilson term, and is required to remove fermion doublers. It vanishes in the continuum $a\to 0$ limit, but its presence complicates the treatment, particularly because the value of the Wilson coefficient r is to be chosen by the practitioner (though $r>0$ is required and $r\leq 1$ is desirable to ensure reflection positivity). We will present results for the two values of $r=1$ and $r=\frac{1}{2}$.

In addition to these, it is necessary to add the following mass (counter)terms to the lattice lagrangian:

$$\begin{aligned}\delta\mathcal{L} = {}& \delta m^2_{f,ij}\Phi^*_{f,i}\Phi_{f,j}+\delta m^2_{a,ij}\Phi^*_{a,i}\Phi_{a,j}+\delta m^2_{s,ij}\Phi^*_{s,i}\Phi_{s,j}+\\ &+\delta M_{f,ij}\psi^\dagger_{f,i}\psi_{f,j}+\delta M_{a,ij}\psi^\dagger_{a,i}\psi_{a,j}+\delta M_{s,ij}\psi^\dagger_{s,i}\psi_{s,j}\,.\end{aligned} \tag{3.5}$$

In principle we should also allow for multiplicative renormalization of the couplings and wave functions by including Z factors for each, but this is not needed at the order of interest.

To write the propagators of the lattice fields it is convenient to define the quantities

$$\tilde{p}^2 \equiv \sum_\mu \frac{4}{a^2}\sin^2\frac{p_\mu a}{2}, \qquad \hat{p}_\mu \equiv \frac{1}{a}\sin p_\mu a\,, \tag{3.6}$$

in terms of which the propagators of the scalar and fermionic fields are

$$\Delta_p = \frac{1}{\tilde{p}^2}, \qquad S_p = \frac{-i\sum_\mu \sigma^\mu\hat{p}_\mu + M_p}{\hat{p}^2+M_p^2}\,. \tag{3.7}$$

Here $M_p=\frac{ar}{2}\tilde{p}^2$ is the momentum dependent effective lattice mass induced by the Wilson term.

We need to do three integrals using these lattice propagators to complete the matching calculation; define

$$\frac{C_{ys}}{4\pi a} \equiv \int_{\mathrm{BZ}}\frac{d^3p}{(2\pi)^3}\left(2\Delta_p+\mathrm{Tr}\,S_pS_p\right), \tag{3.8}$$

$$\frac{C_{yf}}{4\pi} \equiv \int_{\mathrm{BZ}}\frac{d^3p}{(2\pi)^3}\,\Delta_p\left(S_p+S_{-p}\right), \tag{3.9}$$

$$\begin{aligned}\frac{C_{yy}}{16\pi^2} \equiv \int_{BZ}\frac{d^3kd^3p}{(2\pi)^6}\Big(&\Delta_k\Delta_p\Delta_{p+k}-\Delta_k\;\mathrm{Tr}\left(S_p^3(S_{k+p}-S_p)+S^3_{k+p}(S_p-S_{k+p})\right)+\\ &+\Delta_k^2\;\mathrm{Tr}\left(S_pS_{k+p}-\frac{1}{2}(S_p^2+S^2_{k+p})\right)\Big)\,.\end{aligned} \tag{3.10}$$

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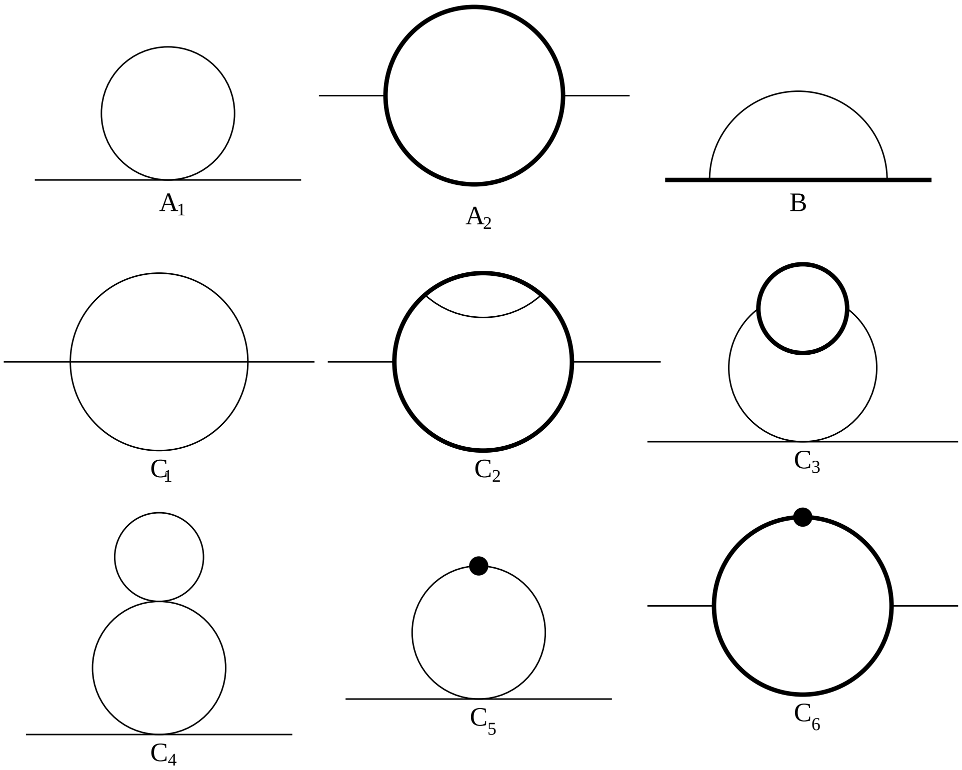

Figure 1: All diagrams needed in scalar and fermionic mass corrections; A are for C_{ys}, B is for C_{yf}, and C are for C_{yy}. Thin lines are scalars, heavy lines are fermions. All of the two loop diagrams have the same combinatorial factor, so the integrands can be added directly to produce an IR finite overall integrand. The heavy dots are one loop mass counterterms, given by the one loop diagrams evaluated at vanishing external momentum.

Here BZ means that the integral is to be taken over a cubic Brillouin zone of extent $(-\pi/a, \pi/a]^3$, that is, each momentum component runs from $-\pi/a$ to π/a. We have included the factors of 4π to imitate the expected behavior that each loop order involves $y^2/4\pi$, and to match notation with previous literature [29]. The diagrams responsible for the contributions are presented in figure 1; the integrations for C_{yy}, the two loop scalar mass correction, have been written in the way presented so that the integrand is absolutely integrable and carries all of its singularities on the three surfaces $k = 0$, $p = 0$, and $p+k = 0$, which is convenient for numerical evaluation.

We can readily verify that, at low momenta, each integrand is well behaved. At leading order for small p, the integrand for C_{ys} behaves as

$$2\Delta(p) + \mathrm{Tr}\, S^2(p) \simeq \frac{2}{p^2} + \frac{\mathrm{Tr}(i\not{p})^2}{p^4} = \frac{2}{p^2} - \frac{2}{p^2} = 0 \qquad (\text{each term is} + \mathrm{O}(p^0))\,. \tag{3.11}$$

Similarly, the integrand of C_{yf} goes as ra/p^2 at small p. The integrand for C_{yy} requires more patience to analyze, but it also proves to be well behaved for $p \to 0$, for $k \to 0$, and for p, k simultaneously taken to zero.

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None of the integrals can be done in closed form (to our knowledge), but all are relatively tractable by quadratures; we find numerically,

$$
\begin{array}{ll}
C_{ys}(r=1) = 6.47060341465275913\,08 & C_{ys}(r=0.5) = 5.057247581039541 \\
C_{yf}(r=1) = 2.29977456857632 & C_{yf}(r=0.5) = 2.22804716126902 \\
C_{yy}(r=1) = 5.425954134(5) & C_{yy}(r=0.5) = 6.8513618(8)\,.
\end{array} \tag{3.12}
$$

In terms of these coefficients, the required renormalizations of the masses are

$$
\begin{aligned}
\delta m^2_{s,ij}(1\text{ loop}) &= \left(-d_{\rm f}\lambda^*_{lmi}\lambda_{lmj} - \frac{1}{2}\xi^*_{ilm}\xi_{jlm}\right)\frac{C_{ys}}{4\pi a}\,, \\
\delta m^2_{f,ij}(1\text{ loop}) &= -\lambda^*_{ilm}\lambda_{jlm}\frac{C_{ys}}{4\pi a}\,, \\
\delta m^2_{a,ij}(1\text{ loop}) &= -\lambda^*_{lim}\lambda_{ljm}\frac{C_{ys}}{4\pi a}\,,
\end{aligned} \tag{3.13}
$$

for the scalar masses at one loop,

$$
\begin{aligned}
\delta M_{s,ij} &= \left(d_{\rm f}\lambda^*_{lmi}\lambda_{lmj} + \frac{1}{2}\xi^*_{ilm}\xi_{jlm}\right)\frac{C_{yf}}{4\pi}\,, \\
\delta M_{f,ij} &= \lambda^*_{ilm}\lambda_{jlm}\frac{C_{yf}}{4\pi}\,, \\
\delta M_{a,ij} &= \lambda^*_{lim}\lambda_{ljm}\frac{C_{yf}}{4\pi}\,,
\end{aligned} \tag{3.14}
$$

for the fermions, and

$$
\begin{aligned}
\delta m^2_{s,ij}(2\text{ loop}) &= \Big\{+d_{\rm f}\lambda^*_{nmi}\lambda_{nqj}\lambda^*_{lqk}\lambda_{lmk} + d_{\rm f}\lambda^*_{mni}\lambda_{qnj}\lambda^*_{qlk}\lambda_{mlk} + \\
&\qquad + d_{\rm f}\xi^*_{qni}\xi_{qmj}\lambda^*_{klm}\lambda_{kln} + \frac{1}{2}\xi^*_{kmi}\xi_{lmj}\xi^*_{knq}\xi_{lnq}\Big\}\frac{C_{yy}}{16\pi^2}\,, \\
\delta m^2_{f,ij}(2\text{ loop}) &= \Big\{\lambda^*_{imn}\lambda_{jqn}\lambda^*_{kql}\lambda_{kml} + \\
&\qquad + d_{\rm f}\lambda^*_{inm}\lambda_{jnq}\lambda^*_{klq}\lambda_{klm} + \frac{1}{2}\lambda^*_{ikl}\lambda_{jkm}\xi^*_{nqm}\xi_{nql}\Big\}\frac{C_{yy}}{16\pi^2}\,, \\
\delta m^2_{a,ij}(2\text{ loop}) &= \Big\{\lambda^*_{min}\lambda_{qjn}\lambda^*_{qkl}\lambda_{mkl} + \\
&\qquad + d_{\rm f}\lambda^*_{nim}\lambda_{njq}\lambda^*_{lkq}\lambda_{lkm} + \frac{1}{2}\lambda^*_{kil}\lambda_{kjm}\xi^*_{nqm}\xi_{nql}\Big\}\frac{C_{yy}}{16\pi^2}\,,
\end{aligned} \tag{3.15}
$$

for the scalar masses at two loops. Here $d_{\rm f}$ is the dimension of the fundamental representation, $d_{\rm f} = N_{\rm c}$ in $\mathrm{SU}(N_{\rm c})$ gauge theory. The full scalar mass counterterm is the sum of the 1 and 2 loop contributions.

This completes the renormalization of the theory at a level which will leave only $\mathrm{O}(a)$ supersymmetry breaking effects.

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4. $N = 2$ SU(N_c) gauge theories with fundamental matter

Now we extend these results to the case where gauge interactions are also present. We will not review here, how gauge fields are put on the lattice; we refer the interested reader to H. Rothe's book [36], which also presents the Feynman rules associated with the pure gauge and gauge/fermion sectors with $1/4 \to C_A/12$ in eq. (15.39) and $2/3(\delta_{AB}\delta_{CD} + \ldots) \to \sum tr(T^A T^B T^C T^D)$ in eq. (15.53b) for the generalization to SU(N_c) (the sum is over all permutations of ABCD).

The added fields are a gauge field A_μ, an adjoint fermionic gaugino χ, and an adjoint scalar field we will write ϕ. In the approach where we obtain a 3-D $N = 2$ supersymmetric theory by dimensional reduction of a 4-D, $N = 1$ SUSY theory, the scalar ϕ is the gauge field component in the direction which was compactified. Besides their gauge interactions, which follow from each species' gauge representation, these fields also introduce new Yukawa and scalar couplings,

$$\begin{aligned}\mathcal{L}(\text{new}) = & \sqrt{2}g \left(\psi^\dagger_{f,i}\, e\, (\chi^A)^*\, T^A\, \Phi_{f,i} + \Phi^*_{f,i}\, T^A\, \psi^\top_{f,i}\, e\, \chi^A + (f \to a) \right) + \\ & + \frac{g^2}{2} \left(\Phi^*_{f,i} T^A \Phi_{f,i} + \Phi^*_{a,i} T^A \Phi_{a,i} \right) \left(\Phi^*_{f,j} T^A \Phi_{f,j} + \Phi^*_{a,j} T^A \Phi_{a,j} \right) + \\ & + g^2 \phi^A \phi^B \left(\Phi^*_{f,i} T^A T^B \Phi_{f,i} + \Phi^*_{a,i} T^A T^B \Phi_{a,i} \right) + \\ & + g\phi^A \left(\psi^\dagger_{f,i} T^A \psi_{f,i} + \psi^\dagger_{a,i} T^A \psi_{a,i} \right) + g\phi^A \chi^\dagger F^A \chi\,, \end{aligned} \tag{4.1}$$

where $F^A_{BC} = -if_{ABC}$ is the generator of SU(N_c) in the adjoint representation and T^A must be treated in context as the group generator in the fundamental representation when between f species, or the antifundamental representation ($-T^*$) when between a species. These interactions, which are local, can be implemented on the lattice in the obvious way. The gauge interaction implementation is not so simple but has been well treated in the literature.

Though the matching calculation as a whole is perfectly IR safe and gauge invariant, individual diagrams are, in general, not. The calculation reduces to determining IR safe combinations of diagrams that are gauge invariant. To this end we define the invariants of the group representation as

$$\begin{aligned} \text{Tr}\left(T^A T^B\right) &\equiv \mathcal{T}_F \delta^{AB} \\ \left(T^A T^A\right)_{ab} &\equiv \mathcal{C}_F \delta_{ab} \\ \text{Tr}\left(F^A F^B\right) &= \left(F^C F^C\right)_{AB} \equiv \mathcal{C}_A \delta_{AB}\,. \end{aligned} \tag{4.2}$$

Here $\mathcal{T}_F \equiv C(N_c)$ is the first invariant of the fundamental (defining) representation of SU(N_c) also called the trace normalization. It is usually chosen to be $\frac{1}{2}$. With that choice the second invariant is $\mathcal{C}_F \equiv C_2(N_c) = (N_c^2 - 1)/2N_c$. $\mathcal{C}_A \equiv C(G) = C_2(G) = N_c$ are the invariants of the adjoint representation of the group. n_f and n_a are the number of fundamental and anti-fundamental multiplets respectively. Each diagram will include a prefactor involving some combination of these invariants, the Yukawa couplings, the dimension of the fundamental representation and the sum $n_f + n_a$. Since the matching calculation is valid for *any* Lie group, we can factorize the diagrams based on this prefactor.

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Two other group theoretic relations are needed to complete the calculation, which we provide here since they appear in the results almost unchanged:

$$T^B T^A T^B = \left(\mathcal{C}_F - \frac{1}{2}\mathcal{C}_A\right) T^A \qquad \text{and} \qquad F^A_{BC} T^B T^C = \frac{1}{2}\mathcal{C}_A T^A \,. \tag{4.3}$$

New counterterms are needed, which we name $\frac{1}{2}\delta m^2_\phi$ and δM_χ; they have the obvious definitions. The new contributions to the one loop mass counterterms turn out to require only one new lattice integral,

$$\frac{C_{gf}}{4\pi} \equiv \int_{BZ} \frac{d^3p}{(2\pi)^3} \left(-\frac{r}{2}\Delta_p + \frac{1-r^2}{4} \left(\frac{M_p}{\hat{p}^2 + M_p^2} \right) \right) \tag{4.4}$$

which is known analytically for $r = 1$ and easily determined for $r = \frac{1}{2}$:

$$C_{gf}(r = 1) = -\frac{\Sigma}{2} \qquad C_{gf}(r = 0.5) = .097938749331668 \tag{4.5}$$

where $\Sigma = 3.17591153562522$ [29]. They are,

$$\begin{aligned}
\left(\delta m^2_{a,f}\right)_{ab} &= -2g^2 \mathcal{C}_F\, \delta_{ab} \frac{C_{ys}}{4\pi a} \,, \\
\left(\delta m^2_\phi\right)_{AB} &= -g^2 \Big((n_f + n_a)\mathcal{T}_F + \mathcal{C}_A \Big) \delta_{AB} \frac{C_{ys}}{4\pi a} \,, \\
\left(\delta M_{a,f}\right)_{ab} &= g^2 \mathcal{C}_F\, \delta_{ab} \frac{C_{gf}}{4\pi} \,, \\
\left(\delta M_\chi\right)_{AB} &= g^2 \mathcal{C}_A \frac{C_{gf}}{4\pi} + g^2 \Big((n_f + n_a)\mathcal{T}_F + \mathcal{C}_A \Big) \delta_{AB} \frac{C_{yf}}{4\pi} \,.
\end{aligned} \tag{4.6}$$

The new two loop counterterms come in many combinations factorized according to the prefactors determined by the group theory (i.e. products of the group invariants). In terms of the nine new integrals defined in the appendix the new counterterms are (obviously $\lambda^*_{ilm}\lambda_{jlm} \to \lambda^*_{lim}\lambda_{ljm}$ for δm^2_a in the second expression)

$$\delta m^2_{s,ij} = g^2 \lambda^*_{lmi}\lambda_{lmj} \mathcal{C}_F d_F \frac{C_g^{\text{sing}}}{16\pi^2} \,, \tag{4.7}$$

$$\begin{aligned}
\left(\delta m^2_{a,f;ij}\right)_{ab} &= g^2 \delta_{ab} \lambda^*_{ilm}\lambda_{jlm} \mathcal{C}_F \frac{C_{g1}^{\text{fund}}}{16\pi^2} + g^4 \delta_{ij}\delta_{ab} \times \\
&\quad \times \left\{ \mathcal{T}_F \mathcal{C}_F (n_f + n_a) \frac{C_{g2}^{\text{fund}}}{16\pi^2} + (\mathcal{C}_F)^2 \left(\frac{C_{g3}^{\text{fund}}}{16\pi^2} - \frac{1}{3}\frac{\Sigma^2}{16\pi^2} \right) + \right. \\
&\quad \left. + \mathcal{C}_F \mathcal{C}_A \left(\frac{C_{g4}^{\text{fund}}}{16\pi^2} + \frac{1}{18}\frac{\Sigma^2}{16\pi^2} \right) - \frac{4}{3}\mathcal{T}_F \mathcal{C}_F \left(\mathcal{C}_F - \frac{1}{6}\mathcal{C}_A \right) \frac{(4\pi\Sigma)}{16\pi^2} \right\} ,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\left(\delta m^2_\phi\right)_{AB} &= g^2 \delta_{AB} \lambda^*_{ijk}\lambda_{ijk} \mathcal{T}_F \frac{C_{g1}^{\text{adj}}}{16\pi^2} + g^4 \delta_{AB} \times \\
&\quad \times \left\{ \mathcal{T}_F \mathcal{C}_F (n_f + n_a) \frac{C_{g2}^{\text{adj}}}{16\pi^2} + \mathcal{T}_F \mathcal{C}_A \frac{C_{g3}^{\text{adj}}}{16\pi^2} + (\mathcal{C}_A)^2 \left(\frac{C_{g4}^{\text{adj}}}{16\pi^2} - \frac{5}{18}\frac{\Sigma^2}{16\pi^2} \right) - \right. \\
&\quad \left. - \frac{4}{3}\mathcal{T}_F \mathcal{C}_A \left(\mathcal{C}_F - \frac{1}{6}\mathcal{C}_A \right) \frac{(4\pi\Sigma)}{16\pi^2} \right\} .
\end{aligned} \tag{4.9}$$

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The last term in each scalar mass correction is from the piece of the 4-point gluon vertex with a group structure that is fully symmetric between the four lines. It is separately gauge invariant and IR finite, and (as shown) can be found in closed form in terms of the constant Σ, defined in eq. (A.6). The group theoretic factor preceding it arises as $\delta_{AB}\delta_{CD}\sum_{\text{perm}} T^A T^B T^C T^D$.

The constants appearing in these expressions are

$$
\begin{aligned}
&C_g^{\text{sing}}(r=1) = 3.588328893(6) && C_g^{\text{sing}}(r=0.5) = 17.8901895(7)\\
&C_{g1}^{\text{fund}}(r=1) = -4.89236097(1) && C_{g1}^{\text{fund}}(r=0.5) = 3.648535(2)\\
&C_{g2}^{\text{fund}}(r=1) = 10.2296763(2) && C_{g2}^{\text{fund}}(r=0.5) = 13.32776(1)\\
&C_{g3}^{\text{fund}}(r=1) = 22.712647140(8) && C_{g3}^{\text{fund}}(r=0.5) = 29.816565(2)\\
&C_{g4}^{\text{fund}}(r=1) = -2.647013(1) && C_{g4}^{\text{fund}}(r=0.5) = 9.051300(7)\\
&C_{g1}^{\text{adj}}(r=1) = 7.75588650(2) && C_{g1}^{\text{adj}}(r=0.5) = 14.526578(3)\\
&C_{g2}^{\text{adj}}(r=1) = 17.536258926(7) && C_{g2}^{\text{adj}}(r=0.5) = 30.769058(1)\\
&C_{g3}^{\text{adj}}(r=1) = -.3347923(2) && C_{g3}^{\text{adj}}(r=0.5) = .76791(1)\\
&C_{g4}^{\text{adj}}(r=1) = 13.0938429(1) && C_{g4}^{\text{adj}}(r=0.5) = 20.658655(8)\,.
\end{aligned}
\tag{4.10}
$$

The number in parentheses is the error in the last digit. It was determined conservatively since the accuracy of these constants is not expected to be a limiting factor in any lattice implementation. This completes the renormalization of the theory at a level which will leave only $\mathrm{O}(a)$ supersymmetry breaking effects.

5. Conclusion

Non-perturbative treatments of supersymmetric theories have been an elusive goal for decades. A recent surge in both interest and progress in this field has created a great amount of excitement in both the theoretical and lattice physics communities. However, these theories are only recently showing results in terms of the study of truly non-perturbative SUSY phenomena. Much of this difficulty has to do with the great technical challenges encountered in the attempt to formulate lattice theories that can reproduce SUSY in the IR. The direction of the field up until this point has been to try to build infrared supersymmetric behavior into the construction of the action. We have argued that, for three dimensional theories, it is feasible and straightforward instead to use the simplest possible action and do the fine tuning of its parameters necessary to obtain infrared supersymmetric behavior. This is possible analytically and does not prove as difficult as one might have feared.

We have performed these tunings for a class of theories displaying $N = 2$ supersymmetry in three dimensions and containing arbitrary numbers of matter multiplets transforming in the fundamental representation of the gauge group. The technique is robust in the sense that it relies only on theoretical principles, like Wilson's effective action formulation, and lattice implementations, like the Wilson action for fermions, that have been rigorously studied for decades. More generally, the entire lattice action is the most simple such construction with the appropriate IR limit, such that the many complications that can arise in such theories are suitably manageable or altogether absent.

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Though we have not done so, it should be very straightforward to extend our results to matter in general representations. This allows, for instance, the extension of our results to SYM theories with higher supersymmetry, in which interesting non-perturbative physics is claimed to exist. For instance, the $N = 8$ SYM theory in three dimensions has been conjectured by Seiberg to possess a non-trivial IR fixed point (an interacting conformal theory) [37]. The theory can be constructed as the $N = 2$ SYM theory with 3 complex matter hyper-multiplets of the $N = 2$ theory all transforming under the adjoint representation of the gauge group. A lattice has recently been constructed in [38] to study this theory by a very different technique.

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A. Diagrams and integrals

In this appendix we enumerate precisely the integrals involved in the two-loop mass correction. All necessary diagrams are given in figure 2. Heavy lines are fermions, light lines are scalars, dotted lines are ghosts and wavy lines are gauge bosons. Heavy dots represent the one-loop mass insertion counter-term, the one-loop diagrams evaluated at zero external momentum. The dark cross in diagram (AX) is the measure counter-term. Diagrams that arise purely as artifacts of the lattice implementation are displayed in a dashed box. Diagrams that do not contribute in Landau gauge are not displayed.

Since it makes structural sense to do so, we have grouped the expression for each two-loop diagram with the analogous mass insertion counter-term. The abbreviations on the LHS correspond to the labels from figure 2. The notation is relatively straightforward: the first letter (F, S or A) labels the field which couple directly to the external scalar line (fermion, scalar or gauge boson respectively), the second, except in the case of the sunset diagrams and (AA6), labels the field coupling secondarily and the number shows whether that secondary coupling is via a 3-point or 4-point vertex (replaced by C when the secondary line "crosses" the diagram). X represents the measure counter-term, a pure artifact of the lattice construction that comes about because the link variable integration measure in the path integral has non-trivial dependence on the gauge fields [36]. G is the Fadeev-Popov ghost, which arises from gauge fixing in direct analogy to the continuum except that the ghost-gauge interaction contains an infinite number of new irrelevant vertices. Of these, only the GGAA vertex will contribute to continuum correlation functions at the perturbative order we require. Its contribution looks like a non-renormalizable mass divergence which is essential to ensure that these divergences cancel in the correlation functions, as is guaranteed by gauge invariance.

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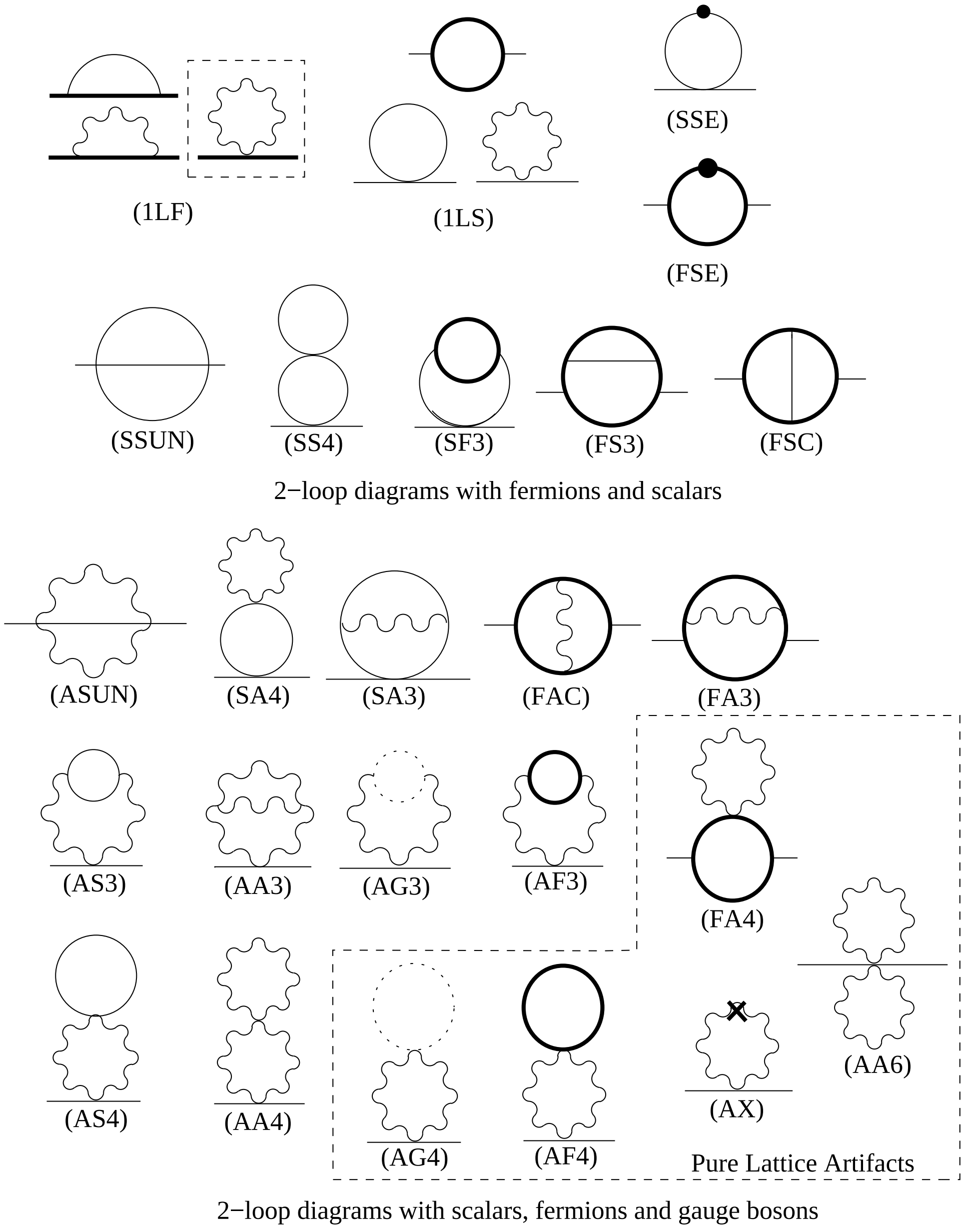

Figure 2: Diagrams needed for the renormalization of SQCD.

The integrands involving only scalars and fermions are (not including $(SS4)$ since its contribution is exactly canceled by its analogous mass counter-term insertion)

$$\begin{aligned}
(SSUN) &= \Delta_k \Delta_p \Delta_{k+p} \\
(SF3) - c.t. &= \Delta_k^2 \ \mathrm{Tr}\left(S_p S_{k+p} - \frac{1}{2}(S_p^2 + S_{k+p}^2) \right) \\
(FS3) - c.t. &= \frac{1}{2}\Delta_k \ \mathrm{Tr}\left(S_p^3 (S_{k+p} - S_p) + S_{k+p}^3 (S_p - S_{k+p}) \right) \\
(FSC) &= \Delta_k \ \mathrm{Tr}(S_p^2 S_{k+p}^2) \, .
\end{aligned} \tag{A.1}$$

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For the scalar-gauge and ghost-gauge sectors we need to define the gauge boson propagator on the lattice (Landau gauge is used throughout the calculations):

$$\tilde{\Delta}_k^{\mu\nu} = \Delta_k \left(\delta_{\mu\nu} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right) ,$$

in terms of which they are

$$\begin{aligned}
(SA3) &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\Delta_p\Delta_{k+p}(\Delta_p+\Delta_{k+p})\widetilde{(2p+k)}_\mu\widetilde{(2p+k)}_\nu \\
(SA4)-c.t. &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\Big(\Delta_p^2(\cos(p_\mu)-1)+\Delta_{k+p}^2(\cos(k+p)_\mu-1)\Big)\delta_{\mu\nu} \\
(ASUN) &= \Delta_k\tilde{\Delta}_p^{\mu\nu}\tilde{\Delta}_{k+p}^{\alpha\beta}\cos\left(\frac{k}{2}\right)_\mu\cos\left(\frac{k}{2}\right)_\nu\delta_{\mu\alpha}\delta_{\nu\beta} \\
(AS3) &= \tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\Delta_p\Delta_{k+p}\widetilde{(2p+k)}_\mu\widetilde{(2p+k)}_\alpha\delta_{\nu\beta} \\
(AS4) &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\Big(\Delta_p\cos(p_\mu)+\Delta_{k+p}\cos(k+p)_\mu\Big)\delta_{\mu\alpha}\delta_{\nu\beta} \\
(AG3) &= \tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\Delta_p\Delta_{k+p}\widetilde{(k+p)}_\mu\cos\left(\frac{p}{2}\right)_\mu\tilde{p}_\alpha\cos\left(\frac{k+p}{2}\right)_\alpha\delta_{\nu\beta} \\
(AG4) &= \frac{1}{24}\tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\Big(\Delta_p^2\,\tilde{p}_\mu\tilde{p}_\mu+\Delta_{k+p}^2\widetilde{(k+p)}_\mu\widetilde{(k+p)}_\mu\Big)\delta_{\mu\alpha}\delta_{\nu\beta}\,.
\end{aligned} \tag{A.2}$$

For the fermion-gauge sector it is convenient to define the momentum part of the $\psi\psi A$ and $\psi\psi AA$ vertices as

$$V_p^\mu = \sigma_\mu\cos\left(\frac{p}{2}\right)_\mu - \frac{ir}{2}\tilde{p}_\mu \qquad \text{and} \qquad V_p^{'\mu} = -\frac{i}{2}\sigma_\mu\tilde{p}_\mu + r\cos\left(\frac{p}{2}\right)_\mu \tag{A.3}$$

respectively. In terms of these the integrands take the form

$$\begin{aligned}
(FAC) &= \tilde{\Delta}_k^{\mu\nu}\ \mathrm{Tr}\left(S_p^2V_{2p+k}^\mu S_{k+p}^2V_{2p+k}^\nu\right) \\
(FA3)-c.t. &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\ \mathrm{Tr}\bigg(S_p^3\Big(V_{2p+k}^\mu S_{k+p}V_{2p+k}^\nu-\frac{1}{2}[V_k^\mu S_kV_k^\nu+V_{-k}^\mu S_{-k}V_{-k}^\nu]\Big)+ \\
&\qquad + S_{k+p}^3\Big(V_{2p+k}^\mu S_pV_{2p+k}^\nu-\frac{1}{2}[V_k^\mu S_kV_k^\nu+V_{-k}^\mu S_{-k}V_{-k}^\nu]\Big)\bigg) \\
(FA4)-c.t. &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\ \mathrm{Tr}\left(S_p^3(V_{2p}^{'\mu}-r)+S_{k+p}^3(V_{2k+2p}^{'\mu}-r)\right)\delta_{\mu\nu} \\
(AF3) &= \tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\,\mathrm{Tr}\,S_pV_{2p+k}^\mu S_{k+p}V_{2p+k}^\alpha\,\delta_{\nu\beta} \\
(AF4) &= \frac{1}{2}\tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\,\mathrm{Tr}\left(S_pV_{2p}^{'\mu}+S_{k+p}V_{2k+2p}^{'\mu}\right)\delta_{\mu\alpha}\delta_{\nu\beta}\,.
\end{aligned} \tag{A.4}$$

Finally, for the pure gauge sector, we have

$$\begin{aligned}
(AX) &= \frac{1}{12}\tilde{\Delta}_k^{\mu\nu}\tilde{\Delta}_k^{\alpha\beta}\delta_{\mu\alpha}\delta_{\nu\beta} = \frac{1}{6}\Delta_k^2 \\
(AA3) &= \tilde{\Delta}_k^{\mu\varepsilon}\tilde{\Delta}_k^{\alpha\varepsilon}\tilde{\Delta}_p^{\nu\beta}\tilde{\Delta}_{k+p}^{\lambda\gamma}\times
\end{aligned} \tag{A.5}$$

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$$\times \left\{ -\delta_{\nu\lambda}\widetilde{(2p+k)}_\mu \cos\left(\frac{k}{2}\right)_\nu + \delta_{\mu\lambda}\widetilde{(2k+p)}_\nu \cos\left(\frac{p}{2}\right)_\lambda + \delta_{\mu\nu}\widetilde{(p-k)}_\lambda \cos\left(\frac{k+p}{2}\right)_\mu \right\} \times$$
$$\times \left\{ +\delta_{\beta\gamma}\widetilde{(2p+k)}_\alpha \cos\left(\frac{k}{2}\right)_\beta - \delta_{\alpha\gamma}\widetilde{(2k+p)}_\beta \cos\left(\frac{p}{2}\right)_\gamma - \delta_{\alpha\beta}\widetilde{(p-k)}_\gamma \cos\left(\frac{k+p}{2}\right)_\alpha \right\}.$$

The diagram labelled (AA6) is an artifact of the lattice discretization. Its contribution can be solved for exactly in terms of the complete elliptic integral of the first kind. Borrowing some now standard notation from [29] we define

$$\Sigma \equiv \frac{1}{\pi^2} \int_{\pi/2}^{\pi/2} d^3x \frac{1}{\sum_i \sin^2(x_i)} = 3.17591153562522\,. \tag{A.6}$$

It is

$$(AA6) = -\frac{1}{4}\left[\frac{2}{3} t^A t^A t^B t^B + \frac{1}{3} t^A t^B t^A t^B\right] \int_{k,p} \tilde{\Delta}_k^{\mu\nu} \tilde{\Delta}_p^{\mu\nu}\,.$$

Repeated Lorentz indices are summed. t^A represents the generator of SU(N_c) in either the defining (T^A_{ab}) or adjoint (F^A_{BC}) representation depending on the transformation properties of the external scalar in question. Appropriate external indices for the product of t's is implied. The group factor in square brackets reduces with the standard relations to

$$\begin{array}{lcl} & \mathcal{C}_F\left(\mathcal{C}_F - \frac{1}{6}\mathcal{C}_A\right) & \text{for external } \Phi_{a,f} \\ \text{or} & \frac{5}{6}\left(\mathcal{C}_A\right)^2 & \text{for external } \phi\,. \end{array}$$

The momentum integral is

$$\int_{k,p} \Delta_k \Delta_p \left(1 + \frac{(\tilde{k}\cdot\tilde{p})^2}{\tilde{k}^2 \tilde{p}^2}\right), \tag{A.7}$$

where $\tilde{k}\cdot\tilde{p} = 4\sum_i \sin(k_i/2)\sin(p_i/2)$. The cross terms from the square are odd in the integration variables and thus integrate to zero in the Brillouin zone. The numerator of the second term can then be rewritten as an average, since the integral is unchanged by $p_1 \leftrightarrow p_2 \leftrightarrow p_3$, and we see that the exact answer

$$\int_{k,p} \tilde{\Delta}_k^{\mu\nu} \tilde{\Delta}_p^{\mu\nu} = \frac{4}{3}\int_k \Delta_k \int_p \Delta_p = \frac{4}{3}\left(\frac{\Sigma}{4\pi}\right)^2\,.$$

Last, though certainly not least, we have the diagram (AA4). We will not reproduce the 4-point gluon vertex rule here but it is written out in [36] (with the changes mentioned in section. 4). The diagram contains two pieces. One involves the group structure $f_{ABE}f_{CDE}$, as in the continuum, and contributes to C_{g4}^{fund} and C_{g4}^{adj}. There is also a pure lattice artifact piece with group structure $\mathrm{Tr}\, T^A T^B T^C T^D$ plus permutations. The contribution of this piece is separately gauge invariant and infrared finite; it can be computed in a way similar to diagram (AA6) above.

In terms of these we need to do 9 numerical integrals. These are composed into IR safe gauge invariant combinations as described in section 4. The numerical factors on the diagrams are a combination of many different factors. Some care must be taken in de-

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termining extra factors of (-1) from diagrams where e commutes across odd numbers of gamma matrices and where odd numbers of the conjugate generator $-T^*$ appear. This occurs most commonly in diagrams with crossed lines. A notable exception is the $-2(SSUN)$ contribution in the first line of eq. (A.9) but this contribution, when lines representing the auxiliary fields are included, also takes the form of a crossed diagram.

For the correction to the singlet scalar the appropriate IR safe gauge invariant combination is (we have changed notation so that analogous mass insertion counterterms are included implicitly by the diagram labels)

$$\frac{C_g^{\rm sing}}{16\pi^2} \equiv 4(SF3) - 6(FS3) + (FSC) - 2(SA3) + (FAC) + 2(FA3) + (FA4)\,. \tag{A.8}$$

For the correction to the fundamental/anti-fundamental scalars they are

$$\begin{aligned}
\frac{C_{g1}^{\rm fund}}{16\pi^2} &\equiv 3(SF3) - 7(FS3) + 4(FSC) - 2(SSUN) - (SA3) + (FA3) + \frac{1}{2}(FA4)\,,\\
\frac{C_{g2}^{\rm fund}}{16\pi^2} &\equiv (SF3) - 4(FS3) + (SSUN) - (AF3) - (AF4) - (AS3) + 2(AS4)\,,\\
\frac{C_{g3}^{\rm fund}}{16\pi^2} &\equiv 2(SF3) - 6(FS3) + 3(SSUN) - (SA3) + 2(ASUN) + 2(FA3) + (FA4)\,,\\
\frac{C_{g4}^{\rm fund}}{16\pi^2} &\equiv (SF3) - 2(FS3) + (FSC) - (SSUN) - (SA3) + (FAC) -\\
&\quad - \frac{1}{2}(ASUN) + 2(FA3) + (FA4) - \frac{1}{2}(AS3) + (AS4) - (AF3) -\\
&\quad - (AF4) + (AG3) - 2(AG4) + \frac{1}{2}(AA3) + \frac{1}{2}(AA4) + 2(AX)\,,
\end{aligned} \tag{A.9}$$

and for the adjoint scalar,

$$\begin{aligned}
\frac{C_{g1}^{\rm adj}}{16\pi^2} &\equiv 4(SF3) - 8(FS3) + 2(FSC)\,,\\
\frac{C_{g2}^{\rm adj}}{16\pi^2} &\equiv 4(SF3) - 6(FS3) - (FSC) + 4(SSUN) - 2(SA3) +\\
&\quad + (FAC) + 2(FA3) + (FA4)\,,\\
\frac{C_{g3}^{\rm adj}}{16\pi^2} &\equiv -4(FS3) + \frac{5}{2}(FSC) - (SSUN) - \frac{1}{2}(FAC) -\\
&\quad - (AF3) - (AF4) - (AS3) + 2(AS4)\,,\\
\frac{C_{g4}^{\rm adj}}{16\pi^2} &\equiv -2(FS3) - \frac{1}{2}(FSC) + \frac{3}{2}(ASUN) + \frac{1}{2}(FAC) + 2(FA3) + (FA4) -\\
&\quad - \frac{1}{2}(AS3) + (AS4) - (AF3) - (AF4) + (AG3) - 2(AG4) +\\
&\quad + \frac{1}{2}(AA3) + \frac{1}{2}(AA4) + 2(AX)\,.
\end{aligned} \tag{A.10}$$

These combinations are observed to be perfectly IR safe through numerical analysis. Note that, as mentioned previously, the label $(AA4)$ contains only that part of the 4-point gluon vertex from [36] that reproduces the appropriate expression in the continuum. The other symmetric piece, described in section 4 just after eq. (4.9), has a unique group structure and is handled separately.

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